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AN APPROXIMATE SHELL THEORY FOR UNRESTRICTED ELASTIC DEFORMATIONS

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Abstract—An approximate shell theory is formulated within the framework of the general theory of finite elasticity for unrestricted deformation. A deformation field is constructed throughout the shell based on the solution of the corresponding membrane problem. An additional deformation is then superposed on this "membrane state" and the resulting equations of motion for the final state linearized in the additional displacement. These equations are then "averaged" through the thickness of the shell to yield an approximate shell theory. Details are carried through for the case in which the additional deformation is itself linearized in the thickness variable so that normals to the middle surface remain straight and uniformly extended, but not necessarily normal. The resulting theory is then applied to the problem of a uniformly twisted, extended and inflated cylindrical shell.

1. INTRODUCTION

A FULLY general non-linear bending theory for elastic shells within the framework of three dimensional large elastic deformation theory is not presently available. The main reason for this appears to be the absence of fully general constitutive equations within this framework relating deformation measures to stress measures for the shell. General and exact treatments of the deformation and equilibrium of shells do exist, for example those found in [1, 2], and more recently [3], where within the framework adopted (Cosserat surfaces) general constitutive equations are furnished. However their relation to the general constitutive equations of large elastic deformation theory has not as yet been fully explored except for some examples (i.e. membrane theory, Kirchhoff hypothesis). Other significant partial results also exist for non-linear material behavior. In particular, Wainwright [4] has considered such shells suffering infinitesimal deformation, and Naghdi and Nordgren [5] have constructed a theory under the Kirchhoff hypothesis. The general theory of elastic membranes (from a three dimensional point of view) may be found in [6], and Corneliussen and Shield [7] considered the addition of an infinitesimal deformation to an already finitely deformed membrane but still without bending and transverse stress effects.

In the present paper a "first order" bending theory is formulated for unrestricted deformation and non-linear elastic behavior of thin shells. It is based on the general non-linear theory of elastic membranes and the linear theory of small deformations superposed on an existing finite deformation. The development is concerned with the interior problem only; the question of appropriate boundary conditions will be considered in a later paper. For the most part general tensor notation is used with the added convention that the range of Latin indices is 1, 2, 3, while that of Greek indices is 1, 2. We use freely well established results of finite elasticity and shell theory; detailed derivations of results of the former and of membrane theory may be found in [6] and [8], of the latter in [2] from which much of the notation in the present paper is taken.

2. BASIC EQUATIONS OF FINITE ELASTICITY

We consider here only homogeneous isotropic elastic bodies. Take $\mathbf{G}_i$ and $\mathbf{g}_i$ to be the covariant base vectors in the reference (natural) and current configurations respectively of the body referred to the convected curvilinear coordinates x^k. The components of the respective metric tensors are thus

$$G_{ij} = \mathbf{G}_i \cdot \mathbf{G}_j \qquad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j$$

and

$$G^{ij} = \frac{\operatorname{cof} G_{ij}}{G} \qquad g^{ij} = \frac{\operatorname{cof} g_{ij}}{g}$$
$$\mathbf{G}^i = G^{ij}\mathbf{G}_j \qquad \mathbf{g}^i = g^{ij}\mathbf{g}_j \tag{2.1}$$

with

$$G = \det\|G_{ij}\| \qquad g = \det\|g_{ij}\|.$$

Since the material is isotropic and elastic there is a stored energy function of the form

$$W = W(I_1, I_2, I_3)$$

where

$$I_1 = g_{mn}G^{mn} \qquad I_2 = I_3 g^{mn}G_{mn} \qquad I_3 = g/G. \tag{2.2}$$

The contravariant components of the stress tensor are then given by

$$\sigma^{rs} = \Phi G^{rs} + \Psi g_{mn}(G^{mn}G^{rs} - G^{mr}G^{ns}) + Pg^{rs} \tag{2.3}$$

where

$$\Phi = 2I_3^{-\frac{1}{2}}\frac{\partial W}{\partial I_1} \qquad \Psi = 2I_3^{-\frac{1}{2}}\frac{\partial W}{\partial I_2} \qquad P = 2I_3^{\frac{1}{2}}\frac{\partial W}{\partial I_3}. \tag{2.4}$$

If the material is incompressible as well as elastic then W becomes a function of only I_1 and I_2. In this case P becomes a new independent variable and we add the constraint condition $I_3 \equiv 1$.

On the surface element whose outward normal is $\mathbf{n} = n_k\mathbf{g}^k$ we have the stress vector

$$\boldsymbol{\sigma}(\mathbf{n}) = \sigma^k(\mathbf{n})\mathbf{g}_k = \sigma^{mk}n_m\mathbf{g}_k. \tag{2.5}$$

In particular, on an exposed surface of the body if $\mathbf{s} = s^k\mathbf{g}_k$ represents the applied surface traction there we must have

$$\boldsymbol{\sigma}(\mathbf{n}) = \mathbf{s} \quad \text{or} \quad \sigma^{mk}n_m = s^k. \tag{2.6}$$

As a consequence of the balance of momentum the stress tensor satisfies the equations of motion

$$\sigma^{rs}{}_{|r} + \rho\mathscr{F}^s = \rho f^s \tag{2.7}$$

where the stroke denotes covariant differentiation with respect to the metric g_{ij}, $\rho = I_3^{-\frac{1}{2}}\rho_0$ is the current mass density of the material, $\mathscr{F} = \mathscr{F}^s\mathbf{g}_s$ is the extrinsic body force density and $\mathbf{f} = f^s\mathbf{g}_s$ the acceleration of the material particles.

An additional deformation may now be superposed on the existing one by adding to the motion of each particle an additional displacement $\varepsilon\mathbf{u}$, where ε is an order parameter eventually to be set equal to unity. If the additional displacement is presumed to be small we linearize the perturbed state equations in the added displacement. Thus if ψ^* represents a quantity in the perturbed state whose value is ψ in the current state, we write

$$\psi^* = \psi + \varepsilon\psi' + o(\varepsilon)$$

and set $\varepsilon = 1$. Then we compute

$$\begin{aligned} &\mathbf{g}'_i = \mathbf{u}_{,i} \qquad && g'_{ij} = \mathbf{g}'_i \cdot \mathbf{g}_j + \mathbf{g}_i \cdot \mathbf{g}'_j \\ &g'^{ij} = -g^{im}g^{jn}g'_{mn} \qquad && \mathbf{g}'^i = g'^{ij}\mathbf{g}_j + g^{ij}\mathbf{g}'_j \\ &\qquad g' = gg^{mn}g'_{mn}. \end{aligned} \tag{2.8}$$

Furthermore

$$\begin{aligned} I'_1 &= G^{mn}g'_{mn} \\ I'_2 &= I_3 G_{rs}(g^{rs}g^{mn} - g^{rm}g^{sn})g'_{mn} \\ I'_3 &= g'/G = I_3 g^{mn}g'_{mn} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \Phi' &= \mathscr{A}I'_1 + \mathscr{F}I'_2 + (\mathscr{E} - \tfrac{1}{2}I_3^{-1}\Phi)I'_3 \\ \Psi' &= \mathscr{F}I'_1 + \mathscr{B}I'_2 + (\mathscr{D} - \tfrac{1}{2}I_3^{-1}\Psi)I'_3 \\ P' &= I_3[\mathscr{E}I'_1 + \mathscr{D}I'_2 + (\mathscr{C} + \tfrac{1}{2}I_3^{-2}P)I'_3] \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} &\mathscr{A} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_1^2} \qquad && \mathscr{B} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_2^2} \qquad && \mathscr{C} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_3^2} \\ &\mathscr{D} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_2 \partial I_3} \qquad && \mathscr{E} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_3 \partial I_1} \qquad && \mathscr{F} = 2I_3^{-\frac{1}{2}}\frac{\partial^2 W}{\partial I_1 \partial I_2} \end{aligned} \tag{2.11}$$

and therefore

$$\begin{aligned} \sigma'^{rs} = {} & \Phi' G^{rs} + \Psi' g_{mn}(G^{mn}G^{rs} - G^{mr}G^{ns}) + P'g^{rs} \\ & + [\Psi(G^{mn}G^{rs} - G^{mr}G^{ns}) - Pg^{mr}g^{ns}]g'_{mn}. \end{aligned} \tag{2.12}$$

For an incompressible material we note that $\mathscr{C}$, $\mathscr{D}$ and $\mathscr{E}$ do not occur while P' is a new independent variable and we have the additional constraint condition $I'_3 \equiv 0$.

The equations of motion in the perturbed state are

$$\sigma^{*rs}{}_{|*r} + \rho^*\mathscr{F}^{*s} = \rho^* f^{*s} \tag{2.13}$$

where the covariant differentiation is with respect to the metric g^*_{ij}, and the components of the vectors $\mathscr{F}^* = \mathscr{F}^{*s}\mathbf{g}^*_s$ and $\mathbf{f}^* = f^{*s}\mathbf{g}^*_s$ are referred to the perturbed base vectors. One may now subtract equation (2.7) and linearize to get

$$\sigma'^{rs}{}_{|r}+\sigma^{ks}\left\{{r \atop kr}\right\}'+\sigma^{rk}\left\{{s \atop kr}\right\}'+\rho[(1+\rho'/\rho)\mathscr{F}^{*s}-\mathscr{F}^{s}] = \rho[(1+\rho'/\rho)f^{*s}-f^{s}] \quad (2.14)$$

where

$$\begin{aligned}\left\{{p \atop mn}\right\}' &= \tfrac{1}{2}g^{pq}[g'_{mq,n}+g'_{qn,m}-g'_{mn,q}-2\left\{{k \atop mn}\right\}g'_{kq}] \\ &= \tfrac{1}{2}g^{pq}[g'_{mq|n}+g'_{qn|m}-g'_{mn|q}]\end{aligned} \quad (2.15)$$

are the additions to the Christoffel symbols. Finally since mass elements are conserved we observe that

$$\rho'/\rho = -\tfrac{1}{2}I'_3/I_3. \quad (2.16)$$

3. BASIC EQUATIONS OF ELASTIC MEMBRANE THEORY

Let x^{α} be convected coordinates in the membrane surface with $\mathbf{a}_{\alpha}$ the corresponding base vectors and $a_{\alpha\beta}$ the surface metric coefficients in the current state. In the reference state we denote the base vectors and metric coefficients by $\mathbf{A}_{\alpha}$ and $A_{\alpha\beta}$ respectively. The membrane surface may be thought of as representing the middle surface of a thin shell so that we introduce a third coordinate x^3 whose corresponding base vector in the current state is a unit vector $\mathbf{a}_3$ in the direction $\mathbf{a}_1\times\mathbf{a}_2$. We observe that $|x^3| \le h/2$ where h is the current thickness of the shell. In the reference state we construct the unit vector $\mathbf{A}_3$ in the direction $\mathbf{A}_1\times\mathbf{A}_2$ and assume that the base vector corresponding to x^3 in the reference state is $\lambda^{-1}\mathbf{A}_3$ where $\lambda = \lambda(x^{\alpha})$ may be thought of as a uniform extension ratio through the thickness of the shell. We use the notation ${}^{\circ}\psi$ to denote that the quantity ψ is evaluated on the membrane surface $x^3 = 0$. Thus we find

$$\begin{array}{lll} {}^{\circ}g_{\alpha\beta} = a_{\alpha\beta} & {}^{\circ}g^{\alpha\beta} = a^{\alpha\beta} & {}^{\circ}g_{\alpha3} = {}^{\circ}g^{\alpha3} = 0 \\ & {}^{\circ}g_{33} = {}^{\circ}g^{33} = 1 & {}^{\circ}g = a \\ {}^{\circ}G_{\alpha\beta} = A_{\alpha\beta} & {}^{\circ}G^{\alpha\beta} = A^{\alpha\beta} & {}^{\circ}G_{\alpha3} = {}^{\circ}G^{\alpha3} = 0 \\ {}^{\circ}G_{33} = \lambda^{-2} & {}^{\circ}G^{33} = \lambda^{2} & {}^{\circ}G = \lambda^{-2}A\ . \end{array} \quad (3.1)$$

The Christoffel symbols evaluated on the membrane surface are found to be

$$\begin{array}{lll} {}^{\circ}\left\{{\rho \atop \alpha\beta}\right\} = \Gamma^{\rho}_{\alpha\beta} & {}^{\circ}\left\{{3 \atop \alpha\beta}\right\} = b_{\alpha\beta} & {}^{\circ}\left\{{\rho \atop \alpha3}\right\} = -b^{\rho}_{\alpha} \\ & {}^{\circ}\left\{{\rho \atop 33}\right\} = {}^{\circ}\left\{{3 \atop \alpha3}\right\} = {}^{\circ}\left\{{3 \atop 33}\right\} = 0 & \end{array} \quad (3.2)$$

where $\Gamma^{\rho}_{\alpha\beta}$ denotes the Christoffel symbol associated with the membrane surface metric $a_{\alpha\beta}$, and

$$b^{\rho\sigma} = a^{\sigma\beta}b^{\rho}_{\beta} = a^{\sigma\beta}a^{\rho\alpha}b_{\alpha\beta}$$

with

$$b_{\alpha\beta} = b_{\beta\alpha} = \mathbf{a}_3\,.\,\mathbf{a}_{\alpha,\beta} \quad (3.3)$$

the coefficients of the second fundamental form for the membrane surface in the current configuration.

It now follows that

$$ {}^{\circ}I_1 = a_{\alpha\beta}A^{\alpha\beta}+\lambda^2 \qquad {}^{\circ}I_2 = {}^{\circ}I_3(a^{\alpha\beta}A_{\alpha\beta}+\lambda^{-2}) \qquad {}^{\circ}I_3 = \lambda^2 a/A \tag{3.4} $$

so that

$$ \begin{aligned} {}^{\circ}\sigma^{\alpha\beta} &= {}^{\circ}\Phi A^{\alpha\beta}+{}^{\circ}\Psi[a_{\rho\sigma}(A^{\rho\sigma}A^{\alpha\beta}-A^{\rho\alpha}A^{\sigma\beta})+\lambda^2 A^{\alpha\beta}]+{}^{\circ}Pa^{\alpha\beta} \\ {}^{\circ}\sigma^{33} &= \lambda^2[{}^{\circ}\Phi+{}^{\circ}\Psi a_{\alpha\beta}A^{\alpha\beta}]+{}^{\circ}P \\ {}^{\circ}\sigma^{\alpha 3} &= 0. \end{aligned} \tag{3.5} $$

Next we take ${}^{\circ}\sigma^{33} = 0$ based on the premise that the membrane stresses in a thin shell are much more significant than the transverse normal stress. This furnishes an equation for the determination of λ^2 if the material is compressible, or for ${}^{\circ}P$ if the material is incompressible ($\lambda^2 = A/a$ in this case is determined from the condition ${}^{\circ}I_3 = 1$).

Now let us restrict attention to equilibrium states so that $\mathbf{f}$ vanishes. We write $\mathscr{P} = \mathscr{P}^k\mathbf{a}_k$ for the total extrinsic force measured per unit area of the current membrane surface. Then the equilibrium equations for the membrane may be written as†

$$ \begin{aligned} (h^{\circ}\sigma^{\beta\alpha})_{\|\beta}+\mathscr{P}^{\alpha} &= 0 \\ hb_{\alpha\beta}{}^{\circ}\sigma^{\alpha\beta}+\mathscr{P}^3 &= 0 \end{aligned} \tag{3.6} $$

where the double stroke denotes covariant differentiation with respect to the membrane surface metric $a_{\alpha\beta}$.

If the deformation pattern defined by membrane theory is extended through the shell thickness we obtain what might be called the "membrane state". This may also be characterized by the statement that the middle surface and its normals are preserved under the deformation, the material being uniformly extended along the normals. Then we can write for the membrane state configuration

$$ \mathbf{g}_\alpha = \mu_\alpha^\rho \mathbf{a}_\rho \qquad \mathbf{g}_3 = \mathbf{a}_3 \tag{3.7} $$

where

$$ \mu_\alpha^\rho = \delta_\alpha^\rho - x^3 b_\alpha^\rho \tag{3.8} $$

so that

$$ \begin{aligned} g_{\alpha\beta} = \mu_\alpha^\rho \mu_\beta^\sigma a_{\rho\sigma} \qquad & g^{\alpha\beta} = \overset{-1}{\mu}{}_\rho^\alpha \overset{-1}{\mu}{}_\sigma^\beta a^{\rho\sigma} \\ g_{\alpha 3} = g^{\alpha 3} = 0 \qquad g_{33} = g^{33} = 1 & \qquad g = \mu^2 a \end{aligned} \tag{3.9} $$

where

$$ \begin{aligned} \mu = |\mu_\alpha^\beta| &= 1-x^3 b_\alpha^\alpha+(x^3)^2|b_\alpha^\beta| \\ \overset{-1}{\mu}{}_\beta^\alpha &= \frac{\operatorname{cof}\mu_\alpha^\beta}{\mu}. \end{aligned} \tag{3.10} $$

† These equations follow from the development leading to equation (4.1) of the next section upon dropping all dependence on the thickness variable x^3, or see for example [6], chapter 4.

A similar treatment for the reference state yields

$$\mathbf{G}_\alpha = M_\alpha^\rho \mathbf{A}_\rho + x^3(\lambda^{-1})_{,\alpha}\mathbf{A}_3 \qquad \mathbf{G}_3 = \lambda^{-1}\mathbf{A}_3 \tag{3.11}$$

where

$$\begin{aligned} M_\alpha^\beta &= \delta_\alpha^\beta - x^3 B_\alpha^\beta \\ B^{\rho\sigma} &= A^{\sigma\beta}B_\beta^\rho = A^{\sigma\beta}A^{\rho\alpha}B_{\alpha\beta} \end{aligned} \tag{3.12}$$

and

$$B_{\alpha\beta} = B_{\beta\alpha} = \lambda^{-1}\mathbf{A}_3 \,.\, \mathbf{A}_{\alpha,\beta}. \tag{3.13}$$

Then setting

$$\begin{aligned} M &= |M_\alpha^\beta| = 1 - x^3 B_\alpha^\alpha + (x^3)^2 |B_\alpha^\beta| \\ \overset{-1}{M}{}_\beta^\alpha &= \frac{\operatorname{cof} M_\alpha^\beta}{M} \end{aligned} \tag{3.14}$$

we find

$$\begin{aligned} &G_{\alpha\beta} = M_\alpha^\rho M_\beta^\sigma A_{\rho\sigma} + (x^3)^2(\lambda^{-1})_{,\alpha}(\lambda^{-1})_{,\beta} \qquad G_{\alpha 3} = x^3\lambda^{-1}(\lambda^{-1})_{,\alpha} \\ &G_{33} = \lambda^{-2} \qquad G = \lambda^{-2}M^2 A \end{aligned} \tag{3.15}$$

$$G^{\alpha\beta} = \overset{-1}{M}{}_\rho^\alpha \overset{-1}{M}{}_\sigma^\beta A^{\rho\sigma} \qquad G^{\alpha 3} = -x^3\lambda(\lambda^{-1})_{,\beta}G^{\alpha\beta} \quad G^{33} = \lambda^2[1 + (x^3)^2 G^{\alpha\beta}(\lambda^{-1})_{,\alpha}(\lambda^{-1})_{,\beta}].$$

On the basis of this deformation pattern and the constitutive equations for the material we may now compute the stresses throughout the shell. If one would wish to maintain the membrane state in equilibrium he would of course have to supply body forces in accordance with equation (2.7), namely

$$\rho\mathscr{F}^s = -\sigma^{rs}{}_{|r}. \tag{3.16}$$

4. APPROXIMATE BENDING THEORY

Let us now consider a class of problems for which the loading consists of a prescribed body force density $\mathscr{F}^*$ and tractions $\mathbf{s}^-$ and $\mathbf{s}^+$ applied to the inner (i.e. $x^3 < 0$) and outer ($x^3 > 0$) surfaces of the shell respectively. The shell is either in an equilibrium configuration or is executing small motion in the neighborhood of an equilibrium configuration. In the latter case we may wish to allow the loading to be time dependent in which event the actual loading is replaced with some time independent average loading so that an equilibrium configuration may be defined in whose neighborhood the small motion takes place. We denote by $x^{*\alpha}$ coordinates in the shell middle surface and x^{*3} is distance measured normal to the middle surface. In general an asterisk on a symbol denotes that the value of the quantity represented is to be based on the actual value in the shell referred to the x^*-coordinates. Thus with σ^{*rs} the components of the stress tensor throughout the shell we note that the equations of motion are simply (2.13). The usual averaging procedure across the shell thickness (see [2], Section 5 for example†) then yields the following set of equations:

$$
\begin{aligned}
N^{*\beta\alpha}_{\|*\beta} - b^{*\alpha}_{\beta} Q^{*\beta} + l^{*\alpha} + p^{*\alpha} &= F^{*\alpha} \\
Q^{*\alpha}_{\|*\alpha} + b^{*}_{\alpha\beta} N^{*\alpha\beta} + l^{*3} + p^{*3} &= F^{*3} \\
M^{*\beta\alpha}_{\|*\beta} - Q^{*\alpha} + m^{*\alpha} + r^{*\alpha} &= G^{*\alpha}
\end{aligned}
\tag{4.1}
$$

where

$$
\begin{aligned}
N^{*\beta\alpha} &= \int_{-h^*/2}^{h^*/2} \mu^* \mu^{*\alpha}_{\rho} \sigma^{*\beta\rho} \, dx^{*3} \\
M^{*\beta\alpha} &= \int_{-h^*/2}^{h^*/2} x^{*3} \mu^* \mu^{*\alpha}_{\rho} \sigma^{*\beta\rho} \, dx^{*3} \\
Q^{*\alpha} &= \int_{-h^*/2}^{h^*/2} \mu^* \sigma^{*\alpha 3} \, dx^{*3}
\end{aligned}
$$

$$
\begin{aligned}
p^{*\alpha} &= \int_{-h^*/2}^{h^*/2} \mu^* \mu^{*\alpha}_{\beta} \rho^* \mathscr{F}^{*\beta} \, dx^{*3} \\
p^{*3} &= \int_{-h^*/2}^{h^*/2} \mu^* \rho^* \mathscr{F}^{*3} \, dx^{*3} \\
r^{*\alpha} &= \int_{-h^*/2}^{h^*/2} x^{*3} \mu^* \mu^{*\alpha}_{\beta} \rho^* \mathscr{F}^{*\beta} \, dx^{*3} \\
F^{*\alpha} &= \int_{-h^*/2}^{h^*/2} \mu^* \mu^{*\alpha}_{\beta} \rho^* f^{*\beta} \, dx^{*3} \\
F^{*3} &= \int_{-h^*/2}^{h^*/2} \mu^* \rho^* f^{*3} \, dx^{*3} \\
G^{*\alpha} &= \int_{-h^*/2}^{h^*/2} x^{*3} \mu^* \mu^{*\alpha}_{\beta} \rho^* f^{*\beta} \, dx^{*3}
\end{aligned}
\tag{4.2}
$$

with $\mathbf{l}^* = l^{*k}\,\mathbf{a}^*_k$ and $\mathbf{m}^* = m^{*\alpha}\mathbf{a}^*_\alpha$ the resultant force and couple load measured per unit area of middle surface and due to the surface tractions $\mathbf{s}^+$ and $\mathbf{s}^-$, i.e.

$$
l^{*k} = \left[\frac{\mu^* \mathbf{s} \,.\, \mathbf{a}^{*k}}{\mathbf{n}^* \,.\, \mathbf{a}^*_3}\right]_{x^{*3}=-h^*/2}^{x^{*3}=h^*/2} \qquad m^{*\alpha} = \left[\frac{\mu^* \mathbf{a}^*_3 \times \mathbf{s} \,.\, \mathbf{a}^{*\alpha}}{|\mathbf{n}^* \,.\, \mathbf{a}^*_3|}\right]_{x^{*3}=-h^*/2}^{x^{*3}=h^*/2}
$$

where $\mathbf{n}^*$ is the unit outward normal to the appropriate shell surface.

The corresponding interior membrane problem is obtained by taking the membrane loading to be precisely the total extrinsic load per unit area of the membrane surface based on the geometry of the membrane state, i.e.

$$
\begin{aligned}
\mathscr{P}^{\alpha} &= \int_{-h/2}^{h/2} \mu \mu^{\alpha}_{\beta} \rho \mathscr{F}^{*\beta} \, dx^3 + \left[\frac{\mu \mathbf{s} \,.\, \mathbf{a}^{\alpha}}{\mathbf{n} \,.\, \mathbf{a}_3}\right]_{x^3=-h/2}^{x^3=h/2} \\
\mathscr{P}^{3} &= \int_{-h/2}^{h/2} \mu \rho \mathscr{F}^{*3} \, dx^3 + \left[\frac{\mu \mathbf{s} \,.\, \mathbf{a}_3}{\mathbf{n} \,.\, \mathbf{a}_3}\right]_{x^3=-h/2}^{x^3=h/2}
\end{aligned}
\tag{4.3}
$$

† Although this derivation is valid only for constant thickness shells, it is easily seen that the resulting equations hold also in the more general case.

where the other variables appearing are based on the resulting membrane state configuration. Furthermore, we suppose that the loading and support conditions for the shell may be phrased so that the membrane equilibrium problem has a solution. We note, as a consequence of (3.6), that

$$(h^\circ\sigma^{\beta\alpha})_{\|\beta} = -\mathscr{P}^\alpha \qquad hb_{\alpha\beta}{}^\circ\sigma^{\alpha\beta} = -\mathscr{P}^3 \tag{4.4}$$

and in the associated membrane state configuration we observe that

$$\begin{aligned} \sigma^{\alpha\beta} &= {}^\circ\sigma^{\alpha\beta} + O(x^3) \\ \sigma^{\alpha 3}, \sigma^{33} &= O(x^3) \end{aligned} \tag{4.5}$$

We next superpose on the membrane state the additional displacements which carry the shell from the membrane state configuration to the actual current configuration, and without further reference linearize all subsequent expressions in the components of this additional displacement. Let us therefore write this additional displacement vector in the form

$$\mathbf{u} = u_k\mathbf{a}^k (= u^k\mathbf{a}_k) = {}^\circ\mathbf{u} + x^3\,\delta\mathbf{u} \tag{4.6}$$

where

$$\begin{aligned} {}^\circ\mathbf{u} &= \mathbf{u}|_{x^3=0} = v_\alpha(x^\rho)\mathbf{a}^\alpha + w(x^\rho)\mathbf{a}^3 (= v^\alpha\mathbf{a}_\alpha + w\mathbf{a}_3) \\ \delta\mathbf{u} &= \xi_\alpha\mathbf{a}^\alpha + \lambda'\mathbf{a}^3 (= \xi^\alpha\mathbf{a}_\alpha + \lambda'\mathbf{a}_3) \end{aligned} \tag{4.7}$$

In what follows it will be important to distinguish between the x- and x^*-coordinate systems in the current configuration; the former are convected coordinates defined as "shell coordinates" in the membrane state, while the latter are "shell coordinates" with respect to the current configuration.

In order to facilitate computation we suppose that to within the degree of approximation entailed in the linearization we may assume that the functions ξ_α and λ' depend only on the middle surface coordinates x^α. Thus the middle surface in the membrane state goes into the middle surface in the perturbed state (i.e. the linearized current state) and we can, without loss of generality, take the shell coordinates to be identical to the convected coordinates on the middle surface; that is, for the same material particle on the surface $x^{*3} = x^3 = 0$, we have $x^{*\alpha} \equiv x^\alpha$. Then one readily computes

$$\begin{aligned} \mathbf{a}'_\alpha &= {}^\circ\mathbf{u}_{,\alpha} = (v_{\rho\|\alpha} - b_{\rho\alpha}w)\mathbf{a}^\rho + (w_{,\alpha} + b^\rho_\alpha v_\rho)\mathbf{a}_3 \\ a'_{\alpha\beta} &= 2[v_{(\alpha\|\beta)} - b_{\alpha\beta}w] \\ a' &= 2a[v^\alpha_{\|\alpha} - b^\alpha_\alpha w] \\ \Gamma'^\rho_{\alpha\beta} &= \tfrac{1}{2}a^{\rho\eta}[a'_{\alpha\eta\|\beta} + a'_{\eta\beta\|\alpha} - a'_{\alpha\beta\|\eta}] \\ &= v^\rho_{\|(\alpha\beta)} + [b^\rho_\eta b_{\alpha\beta} - b_{\eta(\alpha}b^\rho_{\beta)}]v^\eta \\ &\quad + a^{\eta\rho}b_{\alpha\beta}w_{,\eta} - 2b^\rho_{(\alpha}w_{,\beta)} - b^\rho_{\alpha\|\beta}w \end{aligned} \tag{4.8}$$

where, in obtaining the last expression, use was made of the Gauss–Mainardi–Codazzi relations

$$b^\rho_{\alpha\|\beta} = b^\rho_{\beta\|\alpha} \qquad R^\eta_{\rho\alpha\beta} = b^\eta_\alpha b_{\beta\rho} - b^\eta_\beta b_{\alpha\rho}$$

where $R^{\eta}_{\rho\alpha\beta}$ is the Riemann–Christoffel tensor, so that

$$v_{\rho\|[\alpha\beta]} = \tfrac{1}{2}R^{\eta}_{\rho\alpha\beta}v_{\eta}$$

We have also used the convention that parentheses enclosing a pair of indices means that the term is to be symmetrized in those indices, while square brackets mean the term is antisymmetrized.

Next we observe that $\mathbf{a}^*_3$ must be the unit normal to the current middle surface hence

$$\begin{aligned}\mathbf{a}'_3 &= a^{-\frac{1}{2}}[\mathbf{a}'_1 \times \mathbf{a}_2 + \mathbf{a}_1 \times \mathbf{a}'_2] - (\tfrac{1}{2}a'/a)\mathbf{a}_3 \\ &= -(b^{\rho}_{\alpha}v_{\rho} + w_{,\alpha})\mathbf{a}^{\alpha}.\end{aligned}$$

Then

$$\begin{aligned}b'_{\alpha\beta} &= \mathbf{a}'_{\alpha,\beta}\cdot\mathbf{a}_3 + \mathbf{a}_{\alpha,\beta}\cdot\mathbf{a}'_3 \\ &= w_{\|\alpha\beta} + 2b_{\rho(\alpha}v^{\rho}_{\|\beta)} + b^{\rho}_{\alpha\|\beta}v_{\rho} - b_{\alpha\rho}b^{\rho}_{\beta}w\end{aligned} \tag{4.9}$$

and we note that

$$\begin{aligned}b'^{\beta}_{\alpha} &= a^{\beta\rho}b'_{\alpha\rho} + a'^{\beta\rho}b_{\alpha\rho} \\ &= a^{\beta\rho}w_{\|\alpha\rho} + b^{\beta}_{\eta}v^{\eta}_{\|\alpha} - b^{\eta}_{\alpha}v^{\beta}_{\|\eta} + b^{\beta}_{\alpha\|\rho}v^{\rho} + b^{\rho}_{\alpha}b^{\beta}_{\rho}w.\end{aligned} \tag{4.10}$$

Now the integrals of equations (4.2) are evaluated along the x^{*3} coordinate lines in the perturbed state, for which we write the vector line element in the form

$$\mathrm{d}\mathbf{r}^* = \mathrm{d}x^{*3}\mathbf{a}^*_3 = \mathrm{d}x^{*3}[\mathbf{a}_3 - (b^{\beta}_{\alpha}v_{\beta} + w_{,\alpha})\mathbf{a}^{\alpha}].$$

The same material line element in the membrane state configuration may be represented as

$$\mathrm{d}\mathbf{r} = \mathrm{d}x^k\mathbf{g}_k = \mathrm{d}x^3\mathbf{a}_3 + \mathrm{d}x^{\alpha}\mu^{\rho}_{\alpha}\mathbf{a}_{\rho}.$$

But the difference in these vector line elements is just

$$\begin{aligned}\mathrm{d}\mathbf{r}^* - \mathrm{d}\mathbf{r} = \mathrm{d}\mathbf{u} &= \mathrm{d}x^k\mathbf{u}_{,k} \\ &= [\mathrm{d}x^3u_{3,3} + \mathrm{d}x^{\beta}(b^{\alpha}_{\beta}u_{\alpha} + u_{3,\beta})]\mathbf{a}_3 \\ &\quad + [\mathrm{d}x^3u_{\alpha,3} + \mathrm{d}x^{\beta}(u_{\alpha\|\beta} - b_{\alpha\beta}u_3)]\mathbf{a}^{\alpha}.\end{aligned}$$

Upon comparing these expressions and linearizing, it follows that along the x^{*3} coordinate lines the convected coordinates satisfy

$$\frac{\mathrm{d}x^{\alpha}}{\mathrm{d}x^3} = -\bar{\mu}^{1\alpha}_{\rho}\Xi^{\rho} \quad \text{with} \quad \frac{\mathrm{d}x^{*3}}{\mathrm{d}x^3} = 1 + \lambda'$$

where

$$\Xi^{\rho} = a^{\rho\beta}w_{,\beta} + \xi^{\rho} + b^{\rho}_{\beta}v^{\beta}$$

may be recognized as the relative rotation in the $x^{\rho}x^3$-plane of the normal and middle surface at the middle surface.

Let us now consider integrals of the form

$$J^* = \int_{-h^*/2}^{h^*/2} \psi^*(x^{*k})\,\mathrm{d}x^{*3} \qquad K^* = \int_{-h^*/2}^{h^*/2} x^{*3}\psi^*(x^{*k})\,\mathrm{d}x^{*3}$$

where

$$\psi^*(x^{*k}) = \psi(x^k)+\psi'(x^k)$$

with x^* and x referring to the same material particle. Now along the particular x^{*3} coordinate line $x^{*\alpha} = c^\alpha$ we may write

$$\psi(x^\alpha; x^3) = \psi(c^\alpha; x^3)+x^3\kappa^\alpha_\rho\Xi^\rho\psi_{,\alpha}(c^\alpha; x^3)$$

where

$$x^3\kappa^\alpha_\rho = \int_0^{x^3} \overset{-1}{\mu}{}^\alpha_\rho \, dx^3 (= x^3[\delta^\alpha_\rho+O(x^3)]).$$

Thus we write

$$J^* = J+J' \qquad K^* = K+K'$$

with

$$J = \int_{-h/2}^{h/2} \psi(x^k)\, dx^3 \qquad K = \int_{-h/2}^{h/2} x^3\psi(x^k)\, dx^3$$

$$J' = \lambda' J+\Xi^\rho\int_{-h/2}^{h/2} x^3\kappa^\alpha_\rho\psi_{,\alpha}\, dx^3+\int_{-h/2}^{h/2} \psi'\, dx^3$$

$$K' = 2\lambda' K+\Xi^\rho\int_{-h/2}^{h/2} (x^3)^2\kappa^\alpha_\rho\psi_{,\alpha}\, dx^3+\int_{-h/2}^{h/2} x^3\psi'\, dx^3$$

the integrations being performed along the x^3 coordinate lines in the convected system.

With similar definitions for $N^{\alpha\beta}$, $N'^{\alpha\beta}$, etc., equations (4.11) can be put in the form

$$\begin{aligned} N'^{\beta\alpha}_{\|\beta}-b^\alpha_\beta Q'^\beta+\Gamma'^\beta_{\rho\beta}N^{\rho\alpha}+\Gamma'^\alpha_{\rho\beta}N^{\rho\beta}-b'^\alpha_\beta Q^\beta+p'^\alpha &= F^{*\alpha} \\ Q'^\alpha_{\|\alpha}+b_{\alpha\beta}N'^{\alpha\beta}+\Gamma'^\alpha_{\rho\alpha}Q^\rho+b'_{\alpha\beta}N^{\alpha\beta}+p'^3 &= F^{*3} \\ M'^{\beta\alpha}_{\|\beta}-Q'^\alpha+\Gamma'^\beta_{\rho\beta}M^{\rho\alpha}+\Gamma'^\alpha_{\rho\beta}M^{\rho\beta}+r'^\alpha &= G^{*\alpha} \end{aligned} \tag{4.11}$$

with

$$\begin{aligned} p'^\alpha &= p^{*\alpha}+l^{*\alpha}+N^{\beta\alpha}_{\|\beta}-b^\alpha_\beta Q^\beta \\ &= p^{*\alpha}+l^{*\alpha}+\left[\int_{-h/2}^{h/2} \mu\mu^\alpha_\rho\sigma^{\beta\rho}\, dx^3\right]_{\|\beta}-b^\alpha_\beta\int_{-h/2}^{h/2} \mu\sigma^{\beta 3}\, dx^3 \\ &= [p^{*\alpha}+l^{*\alpha}-\mathscr{P}^\alpha]+\left[\int_{-h/2}^{h/2} (\mu\mu^\alpha_\rho\sigma^{\beta\rho}-{}^\circ\sigma^{\beta\alpha})\, dx^3\right]_{\|\beta} \\ &\quad -b^\alpha_\beta\int_{-h/2}^{h/2} \mu\sigma^{\beta 3}\, dx^3 \end{aligned}$$

$$p'^3 = [p^{*3}+l^{*3}-\mathscr{P}^3]+\left[\int_{-h/2}^{h/2} \mu\sigma^{\alpha 3}\, dx^3\right]_{\|\alpha}+b_{\alpha\beta}\int_{-h/2}^{h/2} (\mu\mu^\beta_\rho\sigma^{\alpha\rho}-{}^\circ\sigma^{\alpha\beta})\, dx^3$$

$$r'^\alpha = r^{*\alpha}+m^{*\alpha}+\left[\int_{-h/2}^{h/2} x^3\mu\mu^\alpha_\rho\sigma^{\beta\rho}\, dx^3\right]_{\|\beta}-\int_{-h/2}^{h/2} \mu\sigma^{\alpha 3}\, dx^3.$$

In view of equations (4.5) we note that each of the remaining integrals above is at least $O(h^3)$, and further because of the way in which the $\mathscr{P}^k$ were defined (equations 4.3) the remaining terms can be expected to be small.

Once the membrane state has been determined we may compute the components of the stress tensor in this configuration from the constitutive equations and thus evaluate $N^{\alpha\beta}$, $M^{\alpha\beta}$ and Q^α. Furthermore, it is assumed that enough is known about the nature of the prescribed loading so that the terms p'^k and r'^α may also be explicitly determined. Next we take $\mathbf{f}^* = \ddot{\mathbf{u}}$ treating the membrane state as an equilibrium configuration. (In the event the actual loading is time dependent as described earlier, the varying part of the load is absorbed in p'^k and r'^α.) Then

$$\mathbf{f}^* = f^{*k}\mathbf{g}_k^* = \ddot{u}^k\mathbf{a}_k = \ddot{u}^\alpha \overset{-1}{\mu}{}^\rho_\alpha \mathbf{g}_\rho + \ddot{u}^3\mathbf{g}_3 \tag{4.12}$$

whence

$$f^{*\alpha} = \overset{-1}{\mu}{}^\alpha_\beta \ddot{u}^\beta \qquad f^{*3} = \ddot{u}^3$$

and we have

$$F^{*k} = \int_{-h/2}^{h/2} \mu\rho\ddot{u}^k\,\mathrm{d}x^3 \qquad G^{*\alpha} = \int_{-h/2}^{h/2} x^3\mu\rho\ddot{u}^\alpha\,\mathrm{d}x^3 \tag{4.13}$$

If we write

$$\overset{k}{A} = \int_{-h/2}^{h/2} (x^3)^k\mu\rho\,\mathrm{d}x^3 \tag{4.14}$$

then

$$\begin{aligned} F^{*\alpha} &= \overset{0}{A}\ddot{v}^\alpha + \overset{1}{A}\ddot{\xi}^\alpha \\ F^{*3} &= \overset{0}{A}\ddot{w} + \overset{1}{A}\ddot{\lambda}' \\ G^{*\alpha} &= \overset{1}{A}\ddot{v}^\alpha + \overset{2}{A}\ddot{\xi}^\alpha \end{aligned} \tag{4.15}$$

The only unevaluated terms remaining in equations (4.11) are those involving $N'^{\alpha\beta}$, $M'^{\alpha\beta}$ and Q'^α. We define the following integrals:

$$\begin{aligned} \overset{k}{A}{}^\alpha_\rho &= \int_{-h/2}^{h/2} (x^3)^k\kappa^\eta_\rho(\mu\sigma^{\alpha3})_{,\eta}\,\mathrm{d}x^3 \\ \overset{k}{A}{}^{\alpha\beta}_\rho &= \int_{-h/2}^{h/2} (x^3)^k\kappa^\eta_\rho(\mu\mu^\beta_\nu\sigma^{\alpha\nu})_{,\eta}\,\mathrm{d}x^3 \\ \overset{k}{B}{}^{\alpha\beta}_\rho &= \int_{-h/2}^{h/2} (x^3)^k\mu\overset{-1}{\mu}{}^\beta_\rho\sigma^{\alpha3}\,\mathrm{d}x^3 \\ \overset{k}{B}{}^{\alpha\beta\eta}_\rho &= \int_{-h/2}^{h/2} (x^3)^k\mu\mu^\eta_\nu\overset{-1}{\mu}{}^\beta_\rho\sigma^{\alpha\nu}\,\mathrm{d}x^3 \\ \overset{k}{B}{}^{\alpha\beta} &= \overset{k}{B}{}^{\alpha\beta\eta}_\eta\left(= \int_{-h/2}^{h/2} (x^3)^k\mu\sigma^{\alpha\beta}\,\mathrm{d}x^3\right) \end{aligned} \tag{4.16}$$

all of which may be evaluated explicitly once the membrane state configuration is determined. Then

$$\begin{aligned} N'^{\alpha\beta} &= N''^{\alpha\beta} + \lambda' N^{\alpha\beta} + \Xi^{\rho} \overset{1}{A}{}^{\alpha\beta}_{\rho} \\ &\quad - (b'^{\eta}_{\rho} + \lambda' b^{\eta}_{\rho})(\delta^{\rho}_{\eta} \overset{1}{B}{}^{\alpha\beta} + \overset{1}{B}{}^{\alpha\beta\rho}_{\eta}) \\ M'^{\alpha\beta} &= M''^{\alpha\beta} + 2\lambda' M^{\alpha\beta} + \Xi^{\rho} \overset{2}{A}{}^{\alpha\beta}_{\rho} \\ &\quad - (b'^{\eta}_{\rho} + \lambda' b^{\eta}_{\rho})(\delta^{\rho}_{\eta} \overset{2}{B}{}^{\alpha\beta} + \overset{2}{B}{}^{\alpha\beta\rho}_{\eta}) \\ Q'^{\alpha} &= Q''^{\alpha} + \lambda' Q^{\alpha} + \Xi^{\rho} \overset{1}{A}{}^{\alpha}_{\rho} - (b'^{\eta}_{\beta} + \lambda' b^{\eta}_{\beta}) \overset{1}{B}{}^{\alpha\beta}_{\eta} \end{aligned} \tag{4.17}$$

where

$$\begin{aligned} N''^{\alpha\beta} &= \int_{-h/2}^{h/2} \mu \mu^{\beta}_{\rho} \sigma'^{\alpha\rho} \, \mathrm{d}x^3 \\ M''^{\alpha\beta} &= \int_{-h/2}^{h/2} x^3 \mu \mu^{\beta}_{\rho} \sigma'^{\alpha\rho} \, \mathrm{d}x^3 \\ Q''^{\alpha} &= \int_{-h/2}^{h/2} \mu \sigma'^{\alpha 3} \, \mathrm{d}x^3 \end{aligned} \tag{4.18}$$

are the only unevaluated terms remaining.

From equations (2.8) and (4.7) we have

$$\begin{aligned} g'_{\alpha\beta} &= \mu^{\rho}_{\alpha}[v_{\rho\|\beta} + x^3 \xi_{\rho\|\beta} - b_{\rho\beta}(w + x^3\lambda')] \\ &\quad + \mu^{\rho}_{\beta}[v_{\rho\|\alpha} + x^3 \xi_{\rho\|\alpha} - b_{\rho\alpha}(w + x^3\lambda')] \\ g'_{\alpha 3} &= \Xi_{\alpha} + x^3 \lambda'_{,\alpha} \\ g'_{33} &= 2\lambda'. \end{aligned} \tag{4.19}$$

We then write equation (2.12) in the form

$$\sigma'^{rs} = \tfrac{1}{2} \mathscr{S}^{rsmn} g'_{mn}$$

where

$$\begin{aligned} \mathscr{S}^{rsmn} = &- \Phi G^{rs} g^{mn} + \Psi (G^{rs} G^{kl} - G^{rk} G^{sl})(2\delta^m_k \delta^n_l - g_{kl} g^{mn}) \\ &+ P(g^{rs} g^{mn} - 2 g^{rm} g^{sn}) + 2\mathscr{A} G^{rs} G^{mn} \\ &+ 2\mathscr{B} I_3 g_{pq} G_{kl} (G^{rs} G^{pq} - G^{rp} G^{sq})(g^{kl} g^{mn} - g^{km} g^{ln}) \\ &+ 2\mathscr{F}[g_{kl} G^{mn} (G^{kl} G^{rs} - G^{kr} G^{ls}) + I_3 G_{kl} G^{rs} (g^{kl} g^{mn} - g^{km} g^{ln})] \\ &+ 2\mathscr{C} I_3^2 g^{rs} g^{mn} + 2\mathscr{E} I_3 (G^{rs} g^{mn} + g^{rs} G^{mn}) \\ &+ 2\mathscr{D}[I_3 g_{kl} g^{mn} (G^{kl} G^{rs} - G^{kr} G^{ls}) \\ &+ I_3^2 G_{kl} g^{rs} (g^{kl} g^{mn} - g^{km} g^{ln})] \end{aligned} \tag{4.20}$$

is determined from the constitutive equations in the membrane state.

Now in order that the results in the limiting case be consistent with membrane theory as described in [7], the value of λ' should be selected so that ${}^\circ\sigma'^{33}$ vanishes. However, this is possible only if ${}^\circ\mathscr{S}^{3333}$ is never zero, a condition which is not always plausible. Indeed, if the stored energy function is linear in the invariants then a simple calculation shows that ${}^\circ\mathscr{S}^{3333} \equiv 0$. Some other condition is therefore needed to fix λ' in such cases, a contingency which apparently was overlooked in [7]. We might note that this difficulty cannot occur in the incompressible case as was demonstrated in [7] where both λ' and P' were eliminated with the condition ${}^\circ I'_3 = 0$ in addition to ${}^\circ\sigma'^{33} = 0$.

To resolve this problem in the compressible case we merely ignore any additional thickness change and take $\lambda' \equiv 0$. Then upon writing

$$\overset{k}{S}{}^{rsmn} = \int_{-h/2}^{h/2} (x^3)^k \mu \mathscr{S}^{rs(mn)}\, dx^3 \tag{4.21}$$

we find

$$\left.\begin{aligned}
N''^{\alpha\beta} &= [\overset{0}{S}{}^{\alpha\beta\eta\nu} - b^\beta_\rho \overset{1}{S}{}^{\alpha\rho\eta\nu} - b^\nu_\rho \overset{1}{S}{}^{\alpha\beta\eta\rho} + b^\beta_\rho b^\nu_\tau \overset{2}{S}{}^{\alpha\rho\eta\tau}](v_{\rho\|\nu} - b_{\eta\nu} w) \\
&\quad + [\overset{1}{S}{}^{\alpha\beta\eta\nu} - b^\beta_\rho \overset{2}{S}{}^{\alpha\rho\eta\nu} - b^\nu_\rho \overset{2}{S}{}^{\alpha\beta\eta\rho} + b^\beta_\rho b^\nu_\tau \overset{3}{S}{}^{\alpha\rho\eta\tau}]\xi_{\eta\|\nu} \\
&\quad + [\overset{0}{S}{}^{\alpha\beta\eta 3} - b^\beta_\rho \overset{1}{S}{}^{\alpha\rho\eta 3}]\Xi_\eta \\
M''^{\alpha\beta} &= \text{same as } N''^{\alpha\beta} \text{ except that each overscript is} \\
&\quad\ \text{increased in value by one} \\
Q''^{\alpha} &= [\overset{0}{S}{}^{\alpha 3\eta\nu} - b^\nu_\rho \overset{1}{S}{}^{\alpha 3\eta\rho}](v_{\eta\|\nu} - b_{\eta\nu} w) \\
&\quad + [\overset{1}{S}{}^{\alpha 3\eta\nu} - b^\nu_\rho \overset{2}{S}{}^{\alpha 3\eta\rho}]\xi_{\eta\|\nu} + \overset{0}{S}{}^{\alpha 3\eta 3}\Xi_\eta.
\end{aligned}\right\} \tag{4.22}$$

If the material is incompressible the assumed existence of a deformed membrane state immediately leads to a contradiction, for on the one hand we are prescribing the deformation off the middle surface so that changes in area elements along a normal are in the ratio μ/M, while on the other hand the normal extension ratio λ is constant. This is possible in an incompressible material only if $b^\alpha_\alpha = B^\alpha_\alpha, |b^\beta_\alpha| = |B^\beta_\alpha|$, a not very interesting case. It is therefore somewhat unrealistic to base the computation of the constitutive functions for the material on the membrane state deformation throughout the shell thickness, and the same argument can be used against incorporating the effects of changes in the transverse normal strain, in effect, λ'. We therefore evaluate the material functions Φ, Ψ, P, $\mathscr{A}$, $\mathscr{B}$, $\mathscr{F}$ only on the shell middle surface and ignore the effect of λ' in $g'_{\alpha\beta}$ and $g'_{\alpha 3}$. However, in order to be consistent with membrane theory as developed in [7], in the limiting case, we retain

$$g'_{33} = 2\lambda' = -2(v^\alpha_{\|\alpha} - b^\alpha_\alpha w) \tag{4.23}$$

computed from the requirement ${}^\circ I'_3 = 0$. (The function P' is then determined so that ${}^\circ\sigma'^{33} = 0$.)

Thus for an incompressible material we write

$$\sigma'^{rs} = \tfrac{1}{2}\mathscr{S}^{rsmn} g'_{mn} + g^{rs} P'$$

where for this case we have

$$\begin{aligned}\mathscr{S}^{rsmn} &= 2\Phi(A/a)g^{rm}g^{sn}+2\Psi[a^{\alpha\beta}A_{\alpha\beta}g^{rm}g^{sn}+G^{rs}G^{mn}-G^{rm}G^{sn}]\\ &\quad+2\mathscr{A}G^{rs}G^{mn}+2\mathscr{B}g_{kl}G_{pq}(G^{kl}G^{rs}-G^{kr}G^{ls})(g^{pq}g^{mn}-g^{pm}g^{qn})\\ &\quad+2\mathscr{F}[G_{pq}G^{rs}(g^{pq}g^{mn}-g^{pm}g^{qn})\\ &\quad+g_{kl}G^{mn}(G^{kl}G^{rs}-G^{kr}G^{ls})]\end{aligned} \tag{4.24}$$

where we have dropped the prescript denoting that the material constitutive functions are evaluated on the middle surface, this being understood in the incompressible case. We take P' independent of x^3 and hence

$$\begin{aligned}P' &= -\tfrac{1}{2}{}^{\circ}\mathscr{S}^{33mn\circ}g'_{mn}\\ &= ({}^{\circ}\mathscr{S}^{3333}a^{\alpha\beta}-{}^{\circ}\mathscr{S}^{33(\alpha\beta)})(v_{\alpha\|\beta}-b_{\alpha\beta}w).\end{aligned}$$

Then

$$\begin{aligned}\sigma'^{rs} &= [\mathscr{S}^{rs(\gamma\beta)}\mu^{\alpha}_{\gamma}-\mathscr{S}^{rs33}+g^{rs}({}^{\circ}\mathscr{S}^{3333}\,a^{\alpha\beta}-{}^{\circ}\mathscr{S}^{33(\alpha\beta)})](v_{\alpha\|\beta}-b_{\alpha\beta}w)\\ &\quad+\mathscr{S}^{rs(\alpha 3)}\Xi_{\alpha}+x^3\mathscr{S}^{rs(\gamma\beta)}\mu^{\alpha}_{\gamma}\zeta_{\alpha\|\beta}.\end{aligned} \tag{4.25}$$

Upon writing

$$\begin{aligned}\overset{k}{T}{}^{rs\alpha\beta} &= ({}^{\circ}\mathscr{S}^{3333}a^{\alpha\beta}-{}^{\circ}\mathscr{S}^{33(\alpha\beta)})\int_{-h/2}^{h/2}(x^3)^k\mu g^{rs}\,dx^3\\ &\quad-a^{\alpha\beta}\int_{-h/2}^{h/2}(x^3)^k\mu\mathscr{S}^{rs33}\,dx^3\end{aligned} \tag{4.26}$$

we find, for an incompressible material, that equations (4.22) must be augmented by adding the term

$$(\overset{0}{T}{}^{\alpha\beta\eta\nu}-b^{\beta}_{\rho}\overset{1}{T}{}^{\alpha\rho\eta\nu})(v_{\eta\|\nu}-b_{\eta\nu}w)$$

to $N''^{\alpha\beta}$ and adding a similar term with the overscripts increased by one to $M''^{\alpha\beta}$. In addition, the term

$$\overset{0}{T}{}^{\alpha 3\eta\nu}(v_{\eta\|\nu}-b_{\eta\nu}w)$$

should be added to the expression for Q''^{α}.

Finally we note that for an incompressible material $\rho = \rho_0$ and $\rho' \equiv 0$, hence the integrals of equation (4.14) become

$$\overset{k}{A} = \begin{cases}\dfrac{2\rho_0}{k+1}\left(\dfrac{h}{2}\right)^{k+1}\left[1+\dfrac{k+1}{k+3}\left(\dfrac{h}{2}\right)^2(b^1_1b^2_2-b^2_1b^1_2)\right], & k = 0,2\\ -\dfrac{2\rho_0}{k+2}\left(\dfrac{h}{2}\right)^{k+2}(b^1_1+b^2_2), & k = 1\end{cases}$$

In both cases, whether the material is compressible or incompressible, when equations (4.12), (4.15), (4.17) and whatever form of (4.22) is appropriate are substituted in equations (4.11), we obtain a system of five linear equations in the middle surface displacement components v^α, w and in the relative rotation components Ξ^α. The limiting case ($h \to 0$) reduces to the membrane theory of [7] (except as noted in the compressible case).

5. UNIFORMLY INFLATED, EXTENDED, AND TWISTED CYLINDRICAL TUBE

As an example we consider here the particular case of a circular cylindrical shell of an incompressible elastic material subjected to uniform internal pressure and to an axial tension and a twisting couple applied to the cylinder ends. The deformation of the shell middle surface, considered as a membrane, is presumed to be as indicated in the figure. In the current configuration, with respect to cylindrical polar coordinates (ρ, θ, y), the membrane is placed at $\rho = c$ and surface coordinates are chosen to be

$$x^1 = x = c\theta \qquad x^2 = y$$

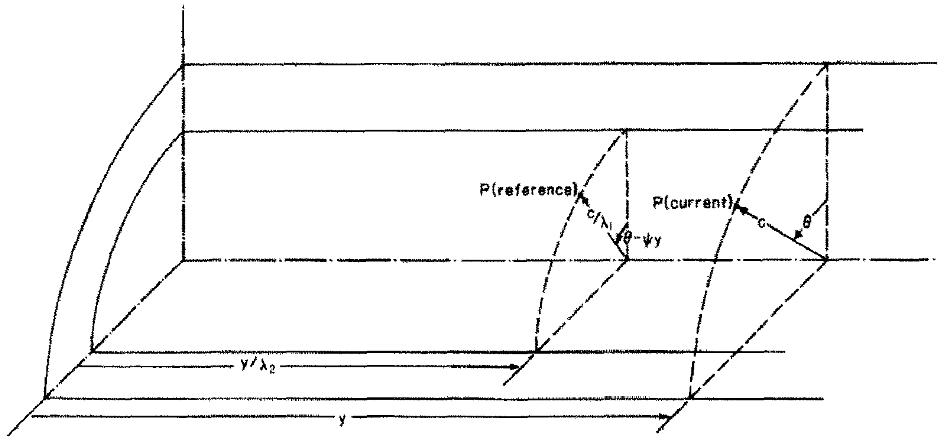

FIG. 1. Presumed middle surface deformation.

Thus one computes easily

$$a_{\alpha\beta} = \delta_{\alpha\beta} \qquad a^{\alpha\beta} = \delta^{\alpha\beta} \qquad a = 1$$

and the only non-vanishing coefficient of the second fundamental form of the surface is

$$b_{11} = b_1^1 = b^{11} = -1/c.$$

Furthermore all Christoffel symbols vanish:

$$\Gamma^\rho_{\alpha\beta} \equiv 0.$$

Now we suppose that the deformation of the shell from the reference configuration to the membrane state is characterized by uniform expansion of the radius and length of the middle surface and of the thickness of the shell (which in the current configuration is assumed to be constant at h) by the factors λ_1, λ_2 and λ respectively, and furthermore the middle surface is twisted at a uniform rate ψ referred to the current length. Thus in the reference state

$$A_{11} = \lambda_1^{-2} \qquad A_{22} = \lambda_2^{-2} + \lambda_1^{-2}c^2\psi^2 \qquad A_{12}(= A_{21}) = -\lambda_1^{-2}c\psi$$
$$A^{11} = \lambda_1^2 + \lambda_2^2 c^2\psi^2 \qquad A^{22} = \lambda_2^2 \qquad A^{12}(= A^{21}) = \lambda_2^2 c\psi \qquad A = \lambda_1^{-2}\lambda_2^{-2}.$$

Because the material is incompressible, we have

$$^\circ I_3 = \lambda^2 A/a = 1$$

hence

$$\lambda = \lambda_1^{-1}\lambda_2^{-1}$$

and

$$B_{11} = -\lambda_2/c \qquad B_{22} = -\lambda_2 c\psi^2 \qquad B_{12}(= B_{21}) = \lambda_2\psi.$$

Next set

$$z = x^3/c \qquad |z| \leq h/2c$$

so that for the membrane state

$$\mu = \mu_1^1 = (\overset{-1}{\mu}{}_1^1)^{-1} = 1+z \qquad \mu_2^2 = \overset{-1}{\mu}{}_2^2 = 1$$

and all other components vanish. Then

$$g = g_{11} = (g^{11})^{-1} = (1+z)^2 \qquad g_{22} = g^{22} = g_{33} = g^{33} = 1$$

with all other components vanishing. Furthermore

$$M = M_1^1 = (\overset{-1}{M}{}_1^1)^{-1} = (1+\lambda_1^2\lambda_2 z) \qquad M_2^2 = \overset{-1}{M}{}_2^2 = 1$$
$$M_1^2 = \overset{-1}{M}{}_1^2 = 0 \qquad M_2^1 = -(1+\lambda_1^2\lambda_2 z)\overset{-1}{M}{}_2^1 = -\lambda_1^2\lambda_2 c\psi z$$

and thus

$$G_{11} = \lambda_1^{-2}(1+\lambda_1^2\lambda_2 z)^2 \qquad G_{22} = \lambda_2^{-2} + \lambda_1^{-2}c^2\psi^2(1+\lambda_1^2\lambda_2 z)^2$$
$$G_{12}(= G_{21}) = -\lambda_1^{-2}c\psi(1+\lambda_1^2\lambda_2 z)^2 \qquad G_{33} = \lambda_1^2\lambda_2^2$$
$$G_{\alpha 3} = 0 \qquad G = (1+\lambda_1^2\lambda_2 z)^2$$
$$G^{11} = \lambda_1^2(1+\lambda_1^2\lambda_2 z)^{-2} + \lambda_2^2 c^2\psi^2 \qquad G^{22} = \lambda_2^2$$
$$G^{12}(= G^{21}) = \lambda_2^2 c\psi \qquad G^{\alpha 3} = 0 \qquad G^{33} = \lambda_1^{-2}\lambda_2^{-2}.$$

For the membrane problem we have

$$^\circ I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_1^{-2}\lambda_2^{-2} + \lambda_2^2 c^2\psi^2 \qquad ^\circ I_2 = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^2\lambda_2^2 + \lambda_1^{-2}c^2\psi^2$$

and therefore

$$^\circ\sigma^{33} = \lambda_1^{-2}\lambda_2^{-2}\Phi + (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^{-2}c^2\psi^2)\Psi + P.$$

With the condition $^\circ\sigma^{33} = 0$ we thus have

$$P = -\lambda_1^{-2}\lambda_2^{-2}\Phi - (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_1^{-2}c^2\psi^2)\Psi$$

and

$$
\begin{aligned}
{}^{\circ}\sigma^{11} &= (\lambda_1^2-\lambda_1^{-2}\lambda_2^{-2}+\lambda_2^2c^2\psi^2)\Phi+(\lambda_1^2\lambda_2^2-\lambda_1^{-2})\Psi \\
{}^{\circ}\sigma^{22} &= (\lambda_2^2-\lambda_1^{-2}\lambda_2^{-2})\Phi+(\lambda_1^2\lambda_2^2-\lambda_2^2-\lambda_1^2c^2\psi^2)\Psi \\
{}^{\circ}\sigma^{12} &= c\psi(\lambda_2^2\Phi+\lambda_1^{-2}\Psi).
\end{aligned}
$$

The constant values of the stress components are of course consistent with the membrane equilibrium equations (for $\mathscr{P}^1 = \mathscr{P}^2 = 0$). In terms of the applied load we have

$$
\begin{aligned}
N &= 2\pi ch{}^{\circ}\sigma^{22} \\
T &= 2\pi c^2h{}^{\circ}\sigma^{12} \\
p(=\mathscr{P}^3) &= (h/c){}^{\circ}\sigma^{11}
\end{aligned}
$$

where N is the axial force extending the cylinder, T is the end couple twisting it, and p the internal pressure inflating it. We note, of course, that $\mathscr{P}^1 = \mathscr{P}^2 = 0$.

In the membrane state we have

$$\sigma^{rs} = (G^{rs}-\lambda_1^{-2}\lambda_2^{-2}g^{rs})\Phi+[g_{mn}(G^{mn}G^{rs}-G^{mr}G^{ns})-(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_1^{-2}c^2\psi^2)g^{rs}]\Psi$$

so that

$$
\begin{aligned}
\sigma^{11} &= [\lambda_1^2(1+\lambda_1^2\lambda_2z)^{-2}+\lambda_2^2c^2\psi^2-\lambda_1^{-2}\lambda_2^{-2}(1+z)^{-2}]\Phi \\
&\quad +[(\lambda_1^2\lambda_2^2+\lambda_2^{-2})(1+\lambda_1^2\lambda_2z)^{-2}+\lambda_1^{-2}c^2\psi^2 \\
&\quad -(\lambda_1^{-2}+\lambda_2^{-2}+\lambda_1^{-2}c^2\psi^2)(1+z)^{-2}]\Psi \\
\sigma^{22} &= [\lambda_2^2-\lambda_1^{-2}\lambda_2^{-2}]\Phi \\
&\quad +[\lambda_1^2\lambda_2^2(1+\lambda_1^2\lambda_2z)^{-2}(1+z)^2-\lambda_2^{-2}-\lambda_1^{-2}c^2\psi^2]\Psi \\
\sigma^{33} &= \{\lambda_2^{-2}[(1+z)^2(1+\lambda_1^2\lambda_2z)^{-2}-1]+\lambda_1^{-2}c^2\psi^2[(1+z)^2-1]\}\Psi \\
\sigma^{12} &= c\psi[\lambda_2^2\Phi+\lambda_1^{-2}\Psi] \\
\sigma^{13} &= \sigma^{23} = 0.
\end{aligned}
$$

We introduce the following direct notation for the components of the additional displacement:

$$
\begin{gathered}
u(x,y) = v^1(x^\alpha) \qquad v(x,y) = v^2(x^\alpha) \qquad w(x,y) = w(x^\alpha) \\
\xi(x,y) = \xi^1(x^\alpha) \qquad \eta(x,y) = \xi^2(x^\alpha).
\end{gathered}
$$

From the incompressibility condition (4.23) we have

$$\lambda' = -(u_x+v_y+w/c)$$

where the subscripts x and y denote partial differentiation. We also denote the relative rotation components of the normals to the middle surface by

$$\Xi(=\xi+w_x-u/c) \qquad \Upsilon(=\eta+w_y).$$

The additional stresses σ'^{rs} given by equation (4.25) are thus of the form

$$\begin{aligned}\sigma'^{rs} = {} & [(1+z)\mathscr{S}^{rs11} - \mathscr{S}^{rs33} + g^{rs}({}^{\circ}\mathscr{S}^{3333} - {}^{\circ}\mathscr{S}^{3311})](u_x + w/c) \\ & + [\mathscr{S}^{rs22} - \mathscr{S}^{rs33} + g^{rs}({}^{\circ}\mathscr{S}^{3333} - {}^{\circ}\mathscr{S}^{3322})]v_y \\ & + \tfrac{1}{2}[(1+z)(\mathscr{S}^{rs12} + \mathscr{S}^{rs21}) - g^{rs}({}^{\circ}\mathscr{S}^{3312} + {}^{\circ}\mathscr{S}^{3321})]u_y \\ & + \tfrac{1}{2}[\mathscr{S}^{rs12} + \mathscr{S}^{rs21} - g^{rs}({}^{\circ}\mathscr{S}^{3312} + {}^{\circ}\mathscr{S}^{3321})]v_x \\ & + \tfrac{1}{2}[\mathscr{S}^{rs13} + \mathscr{S}^{rs31}]\Xi + \tfrac{1}{2}[\mathscr{S}^{rs23} + \mathscr{S}^{rs32}]\Upsilon \\ & + cz\{(1+z)\mathscr{S}^{rs11}\xi_x + \mathscr{S}^{rs22}\eta_y \\ & + \tfrac{1}{2}(\mathscr{S}^{rs12} + \mathscr{S}^{rs21})[(1+z)\xi_y + \eta_x]\}\end{aligned}$$

with the coefficients given in equation (4.24).

Before evaluating the required integrals we expand each integrand in a power series in z, and after integrating, discard all but the lowest order terms in (h/c). (This approximation is consistent with the condition that the additional displacement be linear in z.) Furthermore, while it is conceptually no more difficult to carry through these details for a general incompressible material, a considerable saving in algebraic effort is effected if attention is restricted to materials of the Mooney–Rivlin type, i.e. Φ and Ψ constants so that $\mathscr{A} = \mathscr{B} = \mathscr{F} \equiv 0$. For this case then the equations of motion (4.11), when dynamically uncoupled, are seen to be

$$\begin{aligned}& \beta_5(u_{xx} + w_x/c) + \beta_8 u_{yy} + \beta_{11} u_{xy} \\ & \quad - \beta_9(v_{xx} + v_{yy}) + \beta_7 v_{xy} + \beta_{12} w_y/c \\ & \quad + \tfrac{1}{4}\beta_4 \Xi - \tfrac{1}{2}\beta_9 \Upsilon + \lambda_1^2\lambda_2^2 p'^1/h = \lambda_1^2\lambda_2^2 \rho \ddot{u}\end{aligned}$$

$$\begin{aligned}& -\beta_9(u_{xx} + u_{yy} + w_x/c) + \beta_7 u_{xy} + \beta_8^* v_{xx} \\ & \quad + \beta_1 v_{yy} + \beta_{11} v_{xy} + \beta_2 w_y/c + \lambda_1^2\lambda_2^2 p'^2/h = \lambda_1^2\lambda_2^2 \rho \ddot{v}\end{aligned}$$

$$\begin{aligned}& \beta_6 w_{xx} + \beta_3 w_{yy} + \beta_{10} w_{xy} - \beta_4 w/c^2 - \beta_5 u_x/c \\ & \quad - \beta_{12} u_y/c + \beta_9 v_x/c - \beta_2 v_y/c + \tfrac{1}{4}\beta_4 \Xi_x \\ & \quad + \beta_{13}\Upsilon_y - \tfrac{1}{2}\beta_9(\Xi_y + \Upsilon_x) + \lambda_1^2\lambda_2^2 p'^3/h = \lambda_1^2\lambda_2^2 \rho \ddot{w}\end{aligned}$$

$$\begin{aligned}& -\gamma_1(w_{xxx} + w_{xyy}) - \tfrac{1}{2}\beta_{10}(w_{xxy} + w_{yyy} + 7w_y/c^2) \\ & \quad + \gamma_3 w_x/c^2 + \gamma_5 u_{xx}/c - 3\beta_{12} u_{xy}/c + 3\beta_9 v_{xx}/c - 2\beta_{10} v_{yy}/c \\ & \quad + \gamma_6 v_{xy}/c + \gamma_{14}\Xi_{xx} + \gamma_{10}\Xi_{yy} + \gamma_{11}\Upsilon_{xy} - \tfrac{1}{2}\beta_9(\Xi_{xy} + \Upsilon_{xx}) \\ & \quad - (12c^2/h^2)[\tfrac{1}{4}\beta_4 \Xi/c^2 - \tfrac{1}{2}\beta_9 \Upsilon/c^2] + r''^1 = \lambda_1^2\lambda_2^2 \rho \ddot{\Xi}\end{aligned}$$

$$\begin{aligned}& -\tfrac{1}{2}\beta_{10}(w_{xxx} + w_{xyy} + 3w_x/c^2) - \gamma_2(w_{xxy} + w_{yyy}) \\ & \quad + \gamma_4 w_y/c^2 - \beta_{10} u_{xx}/c - \beta_{12} u_{yy}/c + \gamma_7 u_{xy}/c + \gamma_8 v_{xx}/c + \gamma_9 v_{yy}/c \\ & \quad - \beta_{12} v_{xy}/c + \gamma_{10}\Upsilon_{xx} + \gamma_{12}\Upsilon_{yy} + \gamma_{13}\Xi_{xy} - \tfrac{1}{2}\beta_9(\Xi_{yy} + \Upsilon_{xy}) \\ & \quad + (12c^2/h^2)[\tfrac{1}{2}\beta_9 \Xi/c^2 - \beta_{13}\Upsilon/c^2] + r''^2 = \lambda_1^2\lambda_2^2 \rho \ddot{\Upsilon}\end{aligned}$$

where

$$
\begin{aligned}
\beta_1 &= (3+\lambda_1^2\lambda_2^4)(\Phi+\lambda_1^2\Psi)+3\lambda_2^2c^2\psi^2\Psi \\
\beta_2 &= (3-\lambda_1^4\lambda_2^2-\lambda_1^2\lambda_2^4c^2\psi^2)\Phi+\lambda_2^2(\lambda_1^4\lambda_2^2+1)\Psi \\
\beta_3 &= (\lambda_1^2\lambda_2^4-1)(\Phi+\lambda_1^2\Psi)-\lambda_2^2c^2\psi^2\Psi \\
\beta_4 &= 4(\Phi+\lambda_2^2\Psi) \\
\beta_5 &= (3+\lambda_1^4\lambda_2^2)(\Phi+\lambda_2^2\Psi)+\lambda_1^2\lambda_2^4c^2\psi^2\Phi \\
\beta_6 &= (\lambda_1^4\lambda_2^2-1)(\Phi+\lambda_2^2\Psi)+\lambda_1^2\lambda_2^4c^2\psi^2\Phi \\
\beta_7 &= 3\Phi+(\lambda_1^2+\lambda_2^2+\lambda_1^4\lambda_2^4+\lambda_2^2c^2\psi^2)\Psi \\
\beta_8 &= \lambda_2^2(\lambda_1^2\lambda_2^2\Phi+\Psi) \\
\beta_8^* &= (\lambda_1^2+\lambda_2^2c^2\psi^2)(\lambda_1^2\lambda_2^2\Phi+\Psi) \\
\beta_9 &= 2\lambda_2^2c\psi\Psi \\
\beta_{10} &= 2\lambda_2^2c\psi(\lambda_1^2\lambda_2^2\Phi+\Psi) \\
\beta_{11} &= 2\lambda_2^2c\psi(\lambda_1^2\lambda_2^2\Phi-\Psi) \\
\beta_{12} &= 2\lambda_2^2c\psi\lambda_1^2\lambda_2^2\Phi \\
\beta_{13} &= \Phi+(\lambda_1^2+\lambda_2^2c^2\psi^2)\Psi
\end{aligned}
$$

$$
\begin{aligned}
\gamma_1 &= (1+\lambda_1^4\lambda_2^2+\lambda_1^2\lambda_2^4c^2\psi^2)\Phi+(2\lambda_1^2+\lambda_2^2+\lambda_1^4\lambda_2^4+2\lambda_2^2c^2\psi^2)\Psi \\
\gamma_2 &= (1+\lambda_1^2\lambda_2^4)\Phi+(\lambda_1^2+2\lambda_2^2+\lambda_1^4\lambda_2^4+\lambda_2^2c^2\psi^2)\Psi \\
\gamma_3 &= -2(4+3\lambda_1^4\lambda_2^2-3\lambda_1^2\lambda_2^4c^2\psi^2)\Phi-2(3\lambda_1^2+4\lambda_2^2+3\lambda_1^4\lambda_2^4-2\lambda_1^4\lambda_2+3\lambda_2^2c^2\psi^2)\Psi \\
\gamma_4 &= -(2+\lambda_1^4\lambda_2^2+3\lambda_1^2\lambda_2^4-2\lambda_1^6\lambda_2^3+\lambda_1^2\lambda_2^4c^2\psi^2)\Phi \\
&\quad +(\lambda_1^2-5\lambda_2^2-4\lambda_1^4\lambda_2^4+2\lambda_1^4\lambda_2+\lambda_2^2c^2\psi^2)\Psi \\
\gamma_5 &= -(7+5\lambda_1^4\lambda_2^2+5\lambda_1^2\lambda_2^4c^2\psi^2)\Phi \\
&\quad -(4\lambda_1^2+7\lambda_2^2+5\lambda_1^4\lambda_2^4-4\lambda_1^4\lambda_2+4\lambda_2^2c^2\psi^2)\Psi \\
\gamma_6 &= (-7+3\lambda_1^4\lambda_2^2-2\lambda_1^6\lambda_2^3+3\lambda_1^2\lambda_2^4c^2\psi^2)\Phi \\
&\quad -(8\lambda_1^2+\lambda_2^2+5\lambda_1^4\lambda_2^4-6\lambda_1^4\lambda_2+8\lambda_2^2c^2\psi^2)\Psi \\
\gamma_7 &= -(1+\lambda_1^4\lambda_2^2+2\lambda_1^2\lambda_2^4+\lambda_1^2\lambda_2^4c^2\psi^2)\Phi \\
&\quad +(2\lambda_1^2-3\lambda_2^2-3\lambda_1^4\lambda_2^4+2\lambda_2^2c^2\psi^2)\Psi \\
\gamma_8 &= -2\lambda_1^6\lambda_2^5\Phi-2\lambda_1^4\lambda_2\Psi \\
\gamma_9 &= -(1-\lambda_1^4\lambda_2^2+2\lambda_1^2\lambda_2^4-\lambda_1^2\lambda_2^4c^2\psi^2)\Phi \\
&\quad +(2\lambda_1^2-\lambda_2^2-3\lambda_1^4\lambda_2^4+2\lambda_2^2c^2\psi^2)\Psi \\
\gamma_{10} &= \Phi+(\lambda_1^2+\lambda_1^2-\lambda_1^4\lambda_2^4+\lambda_2^2c^2\psi^2)\Psi \\
\gamma_{11} &= 2\Phi+(2\lambda_1^2+\lambda_2^2+\lambda_1^4\lambda_2^4+2\lambda_2^2c^2\psi^2)\Psi \\
\gamma_{12} &= 3\Phi+(3\lambda_1^2+2\lambda_2^2+3\lambda_2^2c^2\psi^2)\Psi \\
\gamma_{13} &= 2\Phi+(\lambda_1^2+2\lambda_2^2+\lambda_1^4\lambda_2^4+\lambda_2^2c^2\psi^2)\Psi \\
\gamma_{14} &= 3\Phi+(2\lambda_1^2+3\lambda_2^2+2\lambda_2^2c^2\psi^2)\Psi
\end{aligned}
$$

$$r''^1 = \lambda_1^2\lambda_2^2\left\{r'^1 + \frac{h^2}{12}\left[\frac{\partial p'^3}{\partial x} - \frac{2p'^1}{c}\right]\right\}$$

$$r''^2 = \lambda_1^2\lambda_2^2\left\{r'^2 + \frac{h^2}{12}\left[\frac{\partial p'^3}{\partial y} - \frac{p'^2}{c}\right]\right\}.$$

There are several things which may be noted about these equations simply from their form. First, if we consider the limiting case $h/c \to 0$, the relative rotations Ξ and Υ tend to zero identically, and if we also remove the initial twist ($c\psi = 0$), then the first three equations reduce identically to the membrane equations obtained by Corneliussen and Shield [7]. One might also observe that since the equations are homogeneous the membrane solution (for the uniformly extended, inflated and twisted tube) is also a solution which accounts for "first order" bending effects (of which there are none, at least in the interior of the tube). This might have been anticipated because of the high degree of symmetry in the problem. Nevertheless, these equations do represent a refinement of membrane theory in that some bending resistance is allowed in the shell, and in a forthcoming paper they are used to study the stability of the membrane solution in the presence of bending thereby generalizing the results presented in [7].

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Résumé—Une théorie approximative d'enveloppes est formulée dans le cadre de la théorie générale d'élasticité finie pour une déformation sans restriction. Un champ de déformation est construit dans toute l'enveloppe basé sur la solution du problème de membrane correspondant. Une déformation supplémentaire est alors surimposée à cet "état de membrane" et les équations de mouvement qui résultent pour l'état final linéarisées dans le déplacement supplémentaire. Ces équations sont alors prises "en moyenne" au moyen de l'épaisseur de l'enveloppe pour donner une théorie d'enveloppes approximative. Les détails sont suivis pour le cas dans lequel la déformation supplémentaire est elle-même linéarisée dans la variable de l'épaisseur de sorte que les normales à la surface moyenne restent droites et uniformément allongées, mais pas nécessairement normales. La théorie résultante est

Zusammenfassung—Eine annähernde Schalenthoerie wird im Rahmen der allgemeinen Theorie der endlichen Elastizität der unbegränzten Deformierung formuliert. Ein Deformationsfeld wird über die Schale aufgebaut, das auf der Lösung des entsprechenden Membranenproblemes basiert. Eine weitere Deformierung wird dann über diesen 'Membranenzustand' gesetzt und die sich ergebenden Gleichungen des Endzustandes werden in der weiteren Verschiebung linearisiert. Diese Gleichungen werden dann 'auf den Durchschnitt gebracht' durch die Schalendicke und geben die entsprechende Schalenthorie. Einzelheiten werden für den Fall gegeben, wenn die weitere Deformierung selbst linearisiert ist sodass die Dickenvariablen-Normale zur Mitteloberfläche gerade und gleichmässig bleibt aber nicht notwendigerweise normal. Die Theorie wird dann angewandt zur Lösung des Problemes einer gleichmässig verdrehten und vergösserten Zylinderschale.

Абстракт—Приводится формулировка приближенной теории оболочек, в пределах общей теории конечной упругости для бесконечной деформации. Поле деформации, основанное на решении соответствующей задачи мембраны, построено через оболочку. Далее суперпонируется добавочную деформацию к этому "мембранному состоянию" и линеаризуются построенные уравнения движения для остаточного состояния в добавочном перемещении. Тогда эти уравнения "усредняются" по толщине оболочки, чтобы произвести приближенную теорию оболочек. Разработаны детали для случая, в котором добавочная деформация является, сама по себе, линеаризованной по изменяемой толщине так, что нормали к срединной поверхности остаются простыми и равномерно растянутыми, но не всегда перпендикулярными.

Построена теория применена тогда к решению задачи цилиндрической, оболочки, подвергаемой равномерному кручению, растяжению и наполнению.